Chapter 5 Statistical Mechanical Theory of Phase Transition

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5.1

$$E_v = -J \sum_{\langle ij \rangle} s_i s_j \tag{1}$$

When all spins are in the same direction, the total energy become lowest. Now the problem becomes counting the neighbouring pairs. One spin site has $2D$ neighbours. Thus there are $2D \times N/2 = ND$ pairs. Thus $E_0 = -DNJ$.

For a 2D triangular lattice, one spin site has 6 neighbours. Thus the ground state energy is $E_0 = -3NJ$.

5.2

At $T = 0$,

$$\langle M \rangle = \sum_{i=1}^{N} \mu s_i = \pm \mu N \tag{2}$$

The spontaneous magnetization can be either positive or negative, which are equivalent.

5.3

For 1D open Ising chain with N sites,

$$\begin{aligned} Q &= \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} \exp\left(\beta J \sum_{i=1}^{N-1} s_i s_{i+1} \right) \\ &= \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} \prod_{i=1}^{N-1} \exp\left(\beta J s_i s_{i+1} \right) \end{aligned} \tag{3}$$

Instead of considering the value the lattice sites s_i take, we consider the bond values $s_i s_j$. There are $N-1$ independent bonds. Now we prove the equivalency of summation over all site configurations $\{s_i\}$ and summation over all bond configurations $\{s_i s_{i+1}\}$. One site configuration can naturally completely decide one bond configuration. But one bond configuration may corresponding to two site configurations. If we hope to obtain only one site configuration, we have to provide one site spin additionally. However, these two site configurations from one bond configuration have the same energy under zero field.

$$
\begin{aligned}
Q =& 2\sum_{s_1 s_2}\sum_{s_2 s_3}\cdots\sum_{s_{N-1}s_N}\prod_{i=1}^{N-1}\exp(\beta J s_i s_{i+1}) \\
=& 2\prod_{i=1}^{N-1}\left[\sum_{s_i s_{i+1}=\pm 1}\exp\left(\beta J s_i s_{i+1}\right)\right] \\
=& 2\prod_{i=1}^{N-1}\left[2\cosh(\beta J)\right] = 2^N[\cosh(\beta J)]^{N-1}
\end{aligned} \tag{4}
$$

For $N \gg 1$,

$$
Q = [2\cosh(\beta J)]^N. \tag{5}
$$

For 1D Ising chain with N sites under periodic boundary condtion,

$$
\begin{aligned}
Q =& \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}\exp\left(\beta J\sum_{i=1}^{N-1}s_i s_{i+1}\right)\exp(\beta J s_N s_1) \\
=& 2\sum_{s_1 s_2}\sum_{s_2 s_3}\cdots\sum_{s_{N-1}s_N}\prod_{i=1}^{N-1}\exp\left(\beta J s_i s_{i+1}\right)\exp(\beta J s_N s_1) \\
=& 2\sum_{s_1 s_2}\sum_{s_2 s_3}\cdots\sum_{s_{N-1}s_N}\prod_{i=1}^{N-1}\exp\left(\beta J s_i s_{i+1}\right)\frac{1}{2}\{[\exp(\beta J)+\exp(-\beta J)] + s_N s_1[\exp(\beta J)-\exp(-\beta J)]\} \\
=& [2\cosh(\beta J)]^N + \sum_{s_1 s_2}\sum_{s_2 s_3}\cdots\sum_{s_{N-1}s_N}\prod_{i=1}^{N-1}(s_i s_{i+1})\exp\left(\beta J s_i s_{i+1}\right)2\sinh(\beta J) \\
=& [2\cosh(\beta J)]^N + [2\sinh(\beta J)]^N
\end{aligned} \tag{6}
$$

For high temperature $\beta J \to 0$, and large N,

$$
Q = [2\cosh(\beta J)]^N. \tag{7}
$$

5.4

For a lattice gas model

$$
\Xi = \sum_{n_1,\cdots,n_N=0,1}\exp\left[\beta\mu\sum_{i=1}^{N}n_i + \beta\varepsilon\sum_{\langle ij\rangle}n_i n_j\right] \tag{8}
$$

Let $s_i = 2n_i - 1$, and D be the dimensionality. Suppose N is the number of the cells.

$$\begin{aligned}
\Xi &= \sum_{s_1,\cdots,s_N=1,-1} \exp\left[\beta\mu \sum_{i=1}^{N} \frac{s_i+1}{2} + \beta\varepsilon \sum_{\langle ij\rangle} \frac{s_i+1}{2}\frac{s_j+1}{2}\right] \\
&= \sum_{s_1,\cdots,s_N} \exp\left[\frac{\beta\mu}{2}\sum_{i=1}^{N} s_i + \frac{N\beta\mu}{2} + \frac{\beta\varepsilon}{4}\sum_{\langle ij\rangle}(s_is_j + s_i + s_j + 1)\right] \\
&= \sum_{s_1,\cdots,s_N} \exp\left[\frac{\beta\mu}{2}\sum_{i=1}^{N} s_i + \frac{N\beta\mu}{2} + \frac{\beta\varepsilon}{4}\sum_{\langle ij\rangle} s_is_j + \frac{D\beta\varepsilon}{2}\sum_{i=0}^{N} s_i + \frac{ND\beta\varepsilon}{4}\right] \\
&= \sum_{s_1,\cdots,s_N} \exp\left[\left(\frac{N\beta\mu}{2} + \frac{ND\beta\varepsilon}{4}\right) + \frac{(D\varepsilon+\mu)\beta}{2}\sum_{i=1}^{N} s_i + \frac{\beta\varepsilon}{4}\sum_{\langle ij\rangle} s_is_j\right]
\end{aligned} \tag{9}$$

Since a constant coefficient does not change the partition function physically,

$$\Xi = \sum_{s_1,\cdots,s_N} \exp\left[\frac{(D\varepsilon+\mu)\beta}{2}\sum_{i=1}^{N} s_i + \frac{\beta\mu}{4}\sum_{\langle ij\rangle} s_is_j\right]. \tag{10}$$

Comparing to the partition function of a N-site Ising model,

$$H\mu_M = \frac{D\varepsilon+\mu}{2}, \qquad J = \frac{\mu}{4}. \tag{11}$$

5.5

If we make the gravitational field cancel out the $(D\varepsilon+\mu)\beta/2$ term in the equation (10), then the lattice gas model is equivalent to an Ising model with zero external field. In such a case, it is equivalent to treat the occupied cells as gas particles, or to treat the blank cells as gas particles. Thus there is a symmetry in term of the occupation ratio, or the density. Under a certain condition (a certain set of parameters), both a high density phase and a low density phase can be stable. This should be understood as the liquid-gas equilibrium.

5.6

(i) It is easy to see

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{12}$$

Thus from $m = \tanh(\beta Jzm)$ we get

$$\beta Jzm = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right) \tag{13}$$

Thus

$$\beta = \frac{1}{2Jzm}\ln\left(\frac{1+m}{1-m}\right). \tag{14}$$

(ii) By Taylor expansion,

$$\beta = \frac{1}{2Jzm}\left[2m + \frac{2}{3}m^3 + O(m^5)\right] = \frac{1}{Jz}\left[1 + \frac{1}{3}m^2 + O(m^4)\right] \tag{15}$$

$$m^2 \approx 3(\beta Jz - 1) = 6DJ\left(\beta - \frac{1}{2DJ}\right) = 3\left(\frac{\beta - \beta_c}{\beta_c}\right). \tag{16}$$

Thus it is clear to see that the critical exponent β (not the temperature) is $\frac{1}{2}$.
(iii) When $T \to 0$, $\beta \to +\infty$. It is far away from the critical point, thus the expansion above cannot be used. Thus we go back to the equation $m = tanh(\beta Jzm)$.

First we assume $m > 0$. When $\beta \to +\infty$, $\beta Jzm \to +\infty$, then $m = \tanh(\beta Jzm) \to +1$. $m = +1$ does not conflict with our assumption. Then we assume $m < 0$. Similarly we get another solution $m = -1$. Thus the mean field theory gives $m = \pm 1$ at $T = 0$.

5.7

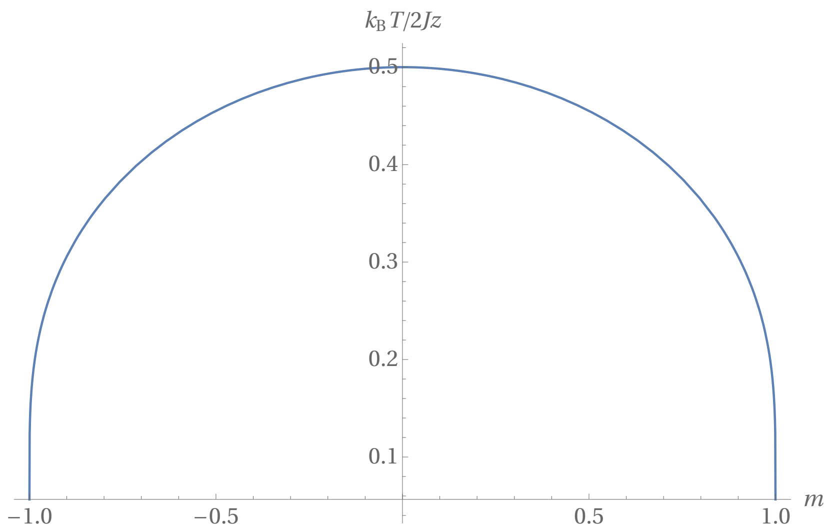

Figure 1: Temperature versus magnetization of the mean field result of the Ising model.

5.8

(i)

$$\langle E\rangle = -H\mu N\langle s_i\rangle - \frac{1}{2}NJz\langle s_i s_j\rangle = -H\mu Nm - \frac{1}{2}NJzm^2. \tag{17}$$

(ii) When $T > T_c$, namely $T > Jz/k_B$, the only solution for m is zero. This prediction for m is correct, since we know that when the temperature higher than T_c the system becomes disordered, and $|M| = 0$. However, the prediction for $\langle E\rangle$ is not correct, being zero when $m = 0$. It should be non-zero because there are spin-spin interations.

5.9

From figure 2 it is obvious that $e^x > 1 + x$ for all real $x \neq 0$. When $x = 0$, $e^x = 1 + x = 1$.

5.10

Because

$$Q \geqslant Q_{MF} \exp(-\beta\langle E - E_{MF}\rangle_{MF}) \tag{18}$$

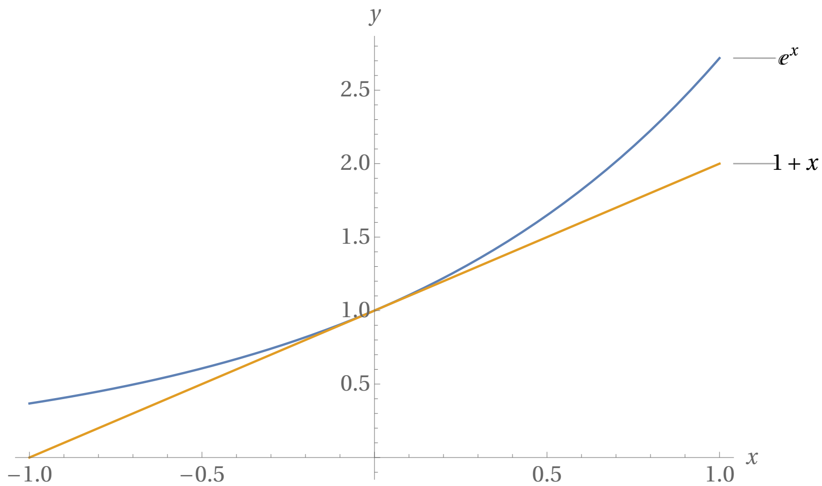

Figure 2: Several functions.

and $A = -\beta^{-1}\ln Q$,

$$A \leqslant -\beta^{-1}\left(\ln Q_{MF} - \beta\langle E - E_{MF}\rangle_{MF}\right) = A_{MF} + \langle E - E_{MF}\rangle_{MF} = A_{PT} \tag{19}$$

where

$$A_{MF} = -\beta^{-1}\ln Q_{MF}. \tag{20}$$

Therefore the first order perturbation free energy A_{PT} is an upper bound to the exact free energy.

5.11

(i)

$$\frac{\partial}{\partial \Delta H} Q_{MF}\exp(-\beta\langle\Delta E\rangle_{MF}) = \left(\frac{\partial Q_{MF}}{\partial \Delta H}\right)\exp(-\beta\langle\Delta E\rangle_{MF}) + Q_{MF}\left(\frac{\partial}{\partial \Delta H}\exp(-\beta\langle\Delta E\rangle_{MF})\right) \tag{21}$$

$$\begin{aligned}\frac{\partial Q_{MF}}{\partial \Delta H} =& N\{2\cosh[\beta\mu(H+\Delta H)]\}^{N-1} 2\sinh[\beta\mu(H+\Delta H)]\beta\mu \\ =& NQ_{MF}\tanh[\beta\mu(H+\Delta H)]\beta\mu \\ =& N\beta\mu Q_{MF}\langle s_1\rangle_{MF}\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial}{\partial \Delta H}\exp(-\beta\langle\Delta E\rangle_{MF}) =& \exp(-\beta\langle\Delta E\rangle_{MF})\frac{\partial(-\beta\langle\Delta E\rangle_{MF})}{\partial \Delta H} \\ =& \exp(-\beta\langle\Delta E\rangle_{MF})\beta N\left[(2Jz\langle s_1\rangle_{MF} - \mu\Delta H)\frac{\partial\langle s_1\rangle}{\partial \Delta H} - \mu\langle s_1\rangle_{MF}\right]\end{aligned} \tag{23}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial \Delta H} Q_{MF}\exp(-\beta\langle\Delta E\rangle_{MF}) =& N\beta Q_{MF}\exp(-\beta\langle\Delta E\rangle_{MF})\left[\mu\langle s_1\rangle_{MF} + (2Jz\langle s_1\rangle_{MF} - \mu\Delta H)\frac{\partial\langle s_1\rangle}{\partial \Delta H} - \mu\langle s_1\rangle_{MF}\right] \\ =& N\beta Q_{MF}\exp(-\beta\langle\Delta E\rangle_{MF})(2Jz\langle s_1\rangle_{MF} - \mu\Delta H)\frac{\partial\langle s_1\rangle}{\partial \Delta H}\end{aligned} \tag{24}$$

Because

$$\frac{\partial}{\partial \Delta H} Q_{MF}\exp(-\beta\langle\Delta E\rangle_{MF}) = 0 \tag{25}$$

$$2Jz\langle s_1\rangle_{MF} = \mu\Delta H. \tag{26}$$

(ii)

$$A_{PT} = -\beta^{-1}\ln Q_{MF} + \langle \Delta E\rangle_{MF} = -\beta^{-1}\ln Q_{MF} - N[Jz\langle s_1\rangle_{MF}^2 - Jz\langle s_1\rangle_{MF}^2] = A_{MF}. \tag{27}$$

Thus $A_{MF} = -\beta^{-1}\ln Q_{MF}$ is the upper bound to the exact free energy.

5.12

(a) Since we hope to find K' and $f(K)$ that satisfy

$$e^{K(s+s')} + e^{-K(s+s')} = f(K)e^{K'ss'}, \tag{28}$$

namely

$$\ln\cosh[K(s+s')] = \ln\frac{f(K)}{2} + K'ss' \tag{29}$$

for every combination of s and s'. Then we get two equations from $s = s'$ and $s = -s'$, respectively.

$$\ln\frac{f(K)}{2} + K' = \ln\cosh 2K, \tag{30}$$

$$\ln\frac{f(K)}{2} - K' = 0. \tag{31}$$

Then the solution is

$$K' = \frac{1}{2}\ln\cosh 2K, \qquad f(K) = 2\cosh^{1/2}(2K). \tag{32}$$

(b) Suppose the logarithm of the partition function can be expressed as

$$\ln Q(N,K) = Ng(K). \tag{33}$$

Because

$$Q(K,N) = [f(K)]^{N/2}Q(K', N/2), \tag{34}$$

or

$$\ln Q(K,N) = \frac{N}{2}\ln f(K) + \ln Q(K', N/2), \tag{35}$$

we get

$$Ng(K) = \frac{N}{2}\ln f(K) + \frac{N}{2}g(K') \tag{36}$$

Thus

$$g(K') = 2g(K) - \ln f(K) \tag{37}$$

Having equation (32)

$$g(K') = 2g(K) - \ln[2\cosh^{1/2}(2K)] \tag{38}$$

(c) From equation (32) it is easy to obtain

$$K = \frac{1}{2}\cosh^{-1}\left(e^{2K'}\right). \tag{39}$$

(d) From equation (38) and (39)

$$g(K) = \frac{1}{2}g(K') + \frac{1}{2}\ln[2\cosh^{1/2}(2K)] = \frac{1}{2}g(K') + \frac{1}{2}K'. \tag{40}$$

(e) Since

$$K' = \frac{1}{2}\ln\cosh(2K) = \frac{1}{2}\ln\left(e^{2K}\frac{1+e^{-4K}}{2}\right) = K + \frac{1}{2}\ln\left(\frac{1+e^{-4K}}{2}\right) \tag{41}$$

Because $K = J/k_BT > 0$, $e^{-4K} < 1$, $\ln[(1+e^{-4K})/2] < \ln 1 = 0$. Then $K' - K < 0$. Therefore $K > K'$.

5.13

(i) Using RG equations (a) and (b): (ii) When using (a) and (b), suppose initially we have an error Δ to our starting $g(K_0)$. Then

$$g^*(K_1) = 2[g(K_0) + \Delta] - \ln f(K_0) = 2[g(K_0)] - \ln f(K_0) + 2\Delta = g(K_1) + 2\Delta \quad (42)$$
$$g^*(K_2) = 2[g(K_1) + 2\Delta] - \ln f(K_1) = g(K_2) + 2^2\Delta \quad (43)$$
$$g^*(K_3) = 2[g(K_2) + 2^2\Delta] - \ln f(K_2) = g(K_3) + 2^3\Delta \quad (44)$$
$$\cdots \quad (45)$$
$$g^*(K_n) = g(K_n) + 2^n\Delta. \quad (46)$$

When we use (c) and (d), similarly, suppose we have an initial error Δ to $g(K_0)$. Then

$$g^*(K_1) = \frac{1}{2}[g(K_0) + \Delta] + \frac{1}{2}\ln 2 + K_0/2 = g(K_1) + \frac{1}{2}\Delta \quad (47)$$
$$g^*(K_2) = \frac{1}{2}[g(K_1) + \frac{1}{2}\Delta] + \frac{1}{2}\ln 2 + K_1/2 = g(K_2) + \frac{1}{2^2}\Delta \quad (48)$$
$$g^*(K_3) = \frac{1}{2}[g(K_2) + \frac{1}{2^2}\Delta] + \frac{1}{2}\ln 2 + K_2/2 = g(K_3) + \frac{1}{2^3}\Delta \quad (49)$$
$$\cdots \quad (50)$$
$$g^*(K_n) = g(K_n) + \frac{1}{2^n}\Delta. \quad (51)$$

5.14

From the group of equations,

$$e^{4K} + e^{-4K} = f(K)\exp(2K_1 + 2K_2 + K_3), \quad (52)$$
$$e^{2K} + e^{-2K} = f(K)\exp(-K_3), \quad (53)$$
$$2 = f(K)\exp(-2K_2 + K_3), \quad (54)$$
$$2 = f(K)\exp(-2K_1 + 2K_2 + K_3), \quad (55)$$

or equivalently,

$$\ln\cosh(4K) = \ln\frac{f(K)}{2} + 2K_1 + 2K_2 + K_3, \quad (56)$$
$$\ln\cosh(2K) = \ln\frac{f(K)}{2} - K_3, \quad (57)$$
$$0 = \ln\frac{f(K)}{2} - 2K_2 + K_3, \quad (58)$$
$$0 = \ln\frac{f(K)}{2} - 2K_1 + 2K_2 + K_3, \quad (59)$$

Combining equations (56), (58), (59) makes

$$\frac{1}{4}\ln\cosh(4K) = \ln\frac{f(K)}{2} + K_3. \quad (60)$$

Then

$$\ln\frac{f(K)}{2} = \frac{1}{2}\ln\cosh(2K) + \frac{1}{8}\ln\cosh(4K), \quad (61)$$
$$K_3 = \frac{1}{8}\ln\cosh(4K) - \frac{1}{2}\ln\cosh(2K). \quad (62)$$

K	$g_{RG}(K)$	$g_{Exact}(K)$
10.0	10.0	10.000000002061153
9.653426409720028	9.653426409720028	9.653426413842336
9.306852819440056	9.306852819440055	9.30685282768467
8.960279229160085	8.960279229160081	8.960279245649314
8.613705638880113	8.613705638880104	8.61370567185857
8.267132048600141	8.267132048600121	8.267132114557056
7.92055845832017	7.9205584583201265	7.920558590233993
7.573984868040206	7.573984868040101	7.573985131867835
7.227411277760268	7.227411277759989	7.227411805415456
6.880837687480435	6.880837687479598	6.880838742790533
6.534264097201019	6.534264097198232	6.534266207820101
6.187690506923274	6.187690506913245	6.187694728156983
5.8411169166522106	5.841116916614333	5.84112535910181
5.494543326407876	5.494543326260846	5.4945602112358
5.147969736270454	5.147969735691293	5.148003505641199
4.801396146560686	4.801396144261954	4.801463684161768
4.454822558561531	4.454822549402431	4.4549576292020605
4.10824897740483	4.108248940840087	4.108519100439345
3.761675423617939	3.76167527750229	3.762215596700806
3.4151019793102573	3.4151013951343776	3.4161820335314093
3.068528972918852	3.068526636789958	3.070687913584021
2.7219577181836025	2.721948374836368	2.7262709284244937
2.3753934699297927	2.375356099182998	2.3840012063592497
2.0288572453109777	2.028707772495073	2.045997986847577
1.6824330785926718	1.6818352858375287	1.7164157145425365
1.3364565580585992	1.3340668330565126	1.4032276904665284
0.9922613073470282	0.9827251782060518	1.1210468930260833
0.6550453340100678	0.6172578418420904	0.8939012714821537
0.3436029520794297	0.19776555104480598	0.7510524103249323
0.10979505396385585	-0.40741113243418925	0.6991625861260634
0.011959300403773162	-1.519928745832097	0.6932186912884081
0.00014301123081965402	-3.7331476834549586	0.6931471907860515
2.0452211838811944e-08	-8.159442567922074	0.6931471805599455
4.440892098500624e-16	-17.012032316404095	0.6931471805599453
0.0	-34.71721181336814	0.6931471805599453
0.0	-70.12757080729622	0.6931471805599453
0.0	-140.9482887951524	0.6931471805599453
0.0	-282.58972477086473	0.6931471805599453
0.0	-565.8725967222894	0.6931471805599453
0.0	-1132.4383406251386	0.6931471805599453
0.0	-2265.5698284308373	0.6931471805599453

Table 1: RG flow from high K to low K for a 2D Ising model.

$$\therefore\ f(K) = 2\cosh^{1/2}(2K)\cosh^{1/8}(4K). \tag{63}$$

From equations (58) and (60) we get K_2

$$K_2 = \frac{1}{2}\left[\ln\frac{f(K)}{2} + K_3\right] = \frac{1}{8}\ln\cosh(4K). \tag{64}$$

Then from equation (59) we get K_1

$$K_1 = 2K_2 = \frac{1}{4}\ln\cosh(4K). \tag{65}$$

5.15

Once we use the approximation

$$K_1\sum_{\langle ij\rangle}s_is_j + K_2\sum_{\langle\langle lm\rangle\rangle}s_ls_m \approx K'(K_1,K_2)\sum_{\langle ij\rangle}s_is_j \tag{66}$$

and neglect K_3 and K_4,

$$Q(K,N) \approx [f(K)]^{N/2}\sum_{N/2\text{ spins}}\exp\left[K'(K_1,K_2)\sum_{\langle ij\rangle}s_is_j\right] = [f(K)]^{N/2}Q[K'(K_1,K_2),N/2] \tag{67}$$

Then

$$g(K) = N^{-1}\ln Q(K,N) = \frac{1}{2}\ln f(K) + N^{-1}\ln Q[(K'(K_1,K_2),N/2] = \frac{1}{2}\ln f(K) + \frac{1}{2}g(K') \tag{68}$$

If we keep using the expression (63), we get

$$g(K') = g(K) - \frac{1}{2}\ln[2\cosh^{1/2}(2K)\cosh^{1/8}(4K)]. \tag{69}$$

5.16

From figure 3 we can see when $K > K_c$, $K' = \frac{3}{8}\ln\cos(4K) > K$; when $K < K_c$, $K' < K$.

5.17

From

$$K' = \frac{3}{8}\ln\cosh(4K) \tag{70}$$

we get

$$K = \frac{1}{4}\cosh^{-1}\left(e^{3K'/8}\right). \tag{71}$$

From

$$g(K') = 2g(K) - \ln[2(\cosh^{1/2}(2K)\cosh^{1/8}(4K)] \tag{72}$$

we get

$$g(K) = \frac{1}{2}g(K') + \frac{1}{2}\ln[2(\cosh^{1/2}(2K)\cosh^{1/8}(4K)] = \frac{1}{2}g(K') + \frac{1}{2}\ln\left\{2e^{K'/3}\cosh^{1/2}\left[\frac{1}{2}\cosh^{-1}\left(e^{8K'/3}\right)\right]\right\} \tag{73}$$

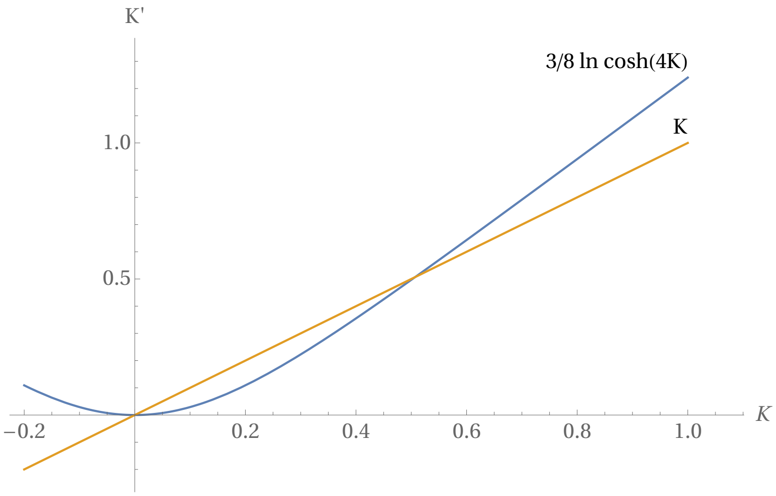

Figure 3: Comparison of K and $3/8 \ln\cosh(4K)$. The non-origin crossing point is at $K = K_c$.

5.18

Because

$$|T - T_c| = \frac{J}{k_B}\left|\frac{1}{K} - \frac{1}{K_c}\right| = \frac{J}{k_B K Kc}|K - K_c| \tag{74}$$

when $T \to T_c$,

$$C \propto |T - T_c|^{-\alpha} \propto |K - K_c|^{-\alpha} \tag{75}$$

Thus $\exists\, \varepsilon > 0$

$$g''(K) = A|K - K_c|^{-\alpha} + O(|K - K_c|^{-\alpha+\varepsilon}) \tag{76}$$

and then

$$g(K) = A'|K - K_c|^{-\alpha+2} + O(|K - K_c|^{-\alpha+\varepsilon+2}). \tag{77}$$

$$g(K) = \frac{1}{2}g(K') + h(K') \tag{78}$$

$$A'|K - K_c|^{-\alpha+2} + O(|K - K_c|^{-\alpha+\varepsilon+2}) = A'|K' - K_c|^{-\alpha+2} + h(K') + O(|K' - K_c|^{-\alpha+\varepsilon+2}) \tag{79}$$

When $T \to T_c$, $K \to K_c$, $K' \to K_c$, $h(K') \to h(K_c)$. Since $|K - K_c| \to 0$, we have $|K' - K_c| \to 0$ and $|K - K'| \to 0$.

$$|K - K_c|^{-\alpha+2} = \frac{1}{2}|K' - K_c|^{-\alpha+2} \tag{80}$$

$$K' \approx K_c + \left.\frac{dK'}{dK}\right|_{K=K_c}(K - K_c) \tag{81}$$

Therefore

$$\left(\left.\frac{dK'}{dK}\right|_{K=K_c}\right)^{-\alpha+2} = 2 \tag{82}$$

$$-\alpha + 2 = \ln 2 \Big/ \ln\left(\left.\frac{dK'}{dK}\right|_{K=K_c}\right) \tag{83}$$

$$\alpha = 2 - \ln 2 \Big/ \ln\left(\left.\frac{dK'}{dK}\right|_{K=K_c}\right) \tag{84}$$

$$\frac{dK'}{dK} = \frac{3}{2}\tanh(4K) \tag{85}$$

Thus

$$\left.\frac{dK'}{dK}\right|_{K=K_c} = 1.4489 \tag{86}$$

$$\alpha = 0.1307. \tag{87}$$

5.19

Suppose the eigenvalue is λ. Then

$$\begin{vmatrix} -\lambda & -\Delta \\ -\Delta & -\lambda \end{vmatrix} = \lambda^2 - \Delta^2 = 0 \tag{88}$$

The solution is

$$\lambda = \pm\Delta \tag{89}$$

Suppose the eigenvector is $(1, k)$. For $\lambda = \pm\Delta$,

$$-\Delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \pm\Delta \begin{pmatrix} 1 \\ k \end{pmatrix} \tag{90}$$

The solution for k is

$$k = \mp 1. \tag{91}$$

Thus the eigenvectors are $(1, -1)$ corresponding to $\lambda = \Delta$ and $(1, 1)$ corresponding to $\lambda = -\Delta$.

5.20

$$\begin{aligned}
&e^{-(\beta/P)(\mathscr{H}_0 - m\mathscr{E})} \\
=&1 - \left(\frac{\beta}{P}\right)(\mathscr{H}_0 - m\mathscr{E}) + \frac{1}{2}\left(\frac{\beta}{P}\right)^2(\mathscr{H}_0 - m\mathscr{E})(\mathscr{H}_0 - m\mathscr{E}) + O\left[\left(\frac{\beta}{P}\right)^3\right] \\
=&1 - \left(\frac{\beta}{P}\right)\mathscr{H}_0 + \left(\frac{\beta}{P}\right)m\mathscr{E} + \frac{1}{2}\left(\frac{\beta}{P}\right)^2\mathscr{H}_0^2 + \frac{1}{2}\left(\frac{\beta}{P}\right)^2 m^2\mathscr{E}^2 - \frac{1}{2}\left(\frac{\beta}{P}\right)^2(\mathscr{H}_0 m\mathscr{E} + m\mathscr{E}\mathscr{H}_0) + O\left[\left(\frac{\beta}{P}\right)^3\right]
\end{aligned} \tag{92}$$

$$\begin{aligned}
&e^{-(\beta/P)}e^{(\mathscr{H}_0 - m\mathscr{E})} \\
=&\left\{1 - \left(\frac{\beta}{P}\right)\mathscr{H}_0 + \frac{1}{2}\left(\frac{\beta}{P}\right)^2\mathscr{H}_0^2 + O\left[\left(\frac{\beta}{P}\right)^3\right]\right\}\left\{1 + \left(\frac{\beta}{P}\right)m\mathscr{E} + \frac{1}{2}\left(\frac{\beta}{P}\right)^2 m^2\mathscr{E}^2 + O\left[\left(\frac{\beta}{P}\right)^3\right]\right\} \\
=&1 - \left(\frac{\beta}{P}\right)\mathscr{H}_0 + \left(\frac{\beta}{P}\right)m\mathscr{E} + \frac{1}{2}\left(\frac{\beta}{P}\right)^2\mathscr{H}_0^2 + \frac{1}{2}\left(\frac{\beta}{P}\right)^2 m^2\mathscr{E}^2 - \left(\frac{\beta}{P}\right)^2 m\mathscr{E}\mathscr{H}_0 + O\left[\left(\frac{\beta}{P}\right)^3\right]
\end{aligned} \tag{93}$$

Thus

$$
\begin{aligned}
&e^{-(\beta/P)(\mathscr{H}_0-m\mathscr{E})}-e^{-(\beta/P)}e^{(\mathscr{H}_0-m\mathscr{E})}\\
=&-\frac{1}{2}\left(\frac{\beta}{P}\right)^2(\mathscr{H}_0 m\mathscr{E}+m\mathscr{E}\mathscr{H}_0)+\left(\frac{\beta}{P}\right)^2 m\mathscr{E}\mathscr{H}_0+O\left[\left(\frac{\beta}{P}\right)^3\right]\\
=&-\frac{1}{2}\left(\frac{\beta}{P}\right)^2(\mathscr{H}_0 m\mathscr{E}-m\mathscr{E}\mathscr{H}_0)+O\left[\left(\frac{\beta}{P}\right)^3\right]\\
=&-\frac{1}{2}\left(\frac{\beta}{P}\right)^2[\mathscr{H}_0, m\mathscr{E}]+O\left[\left(\frac{\beta}{P}\right)^3\right].
\end{aligned}
\tag{94}
$$

5.21

(a) Denote

$$
q_{-1,-1}=\exp(-h+K),\quad q_{-1,+1}=\exp(-K),\quad q_{+1,-1}=\exp(-K),\quad q_{+1,+1}=\exp(h+K). \tag{95}
$$

Thus

$$
q_{s_i,s_{i+1}}=\exp\left[\frac{1}{2}h(s_i+s_{i+1})+Ks_is_{i+1}\right] \tag{96}
$$

$$
\begin{aligned}
\mathrm{Tr}(\mathbf{q}^N)=&\sum_{s_1=\pm1}q^N_{s_1,s_1}=\sum_{s_1=\pm1}\left(\sum_{s_2=\pm1}q_{s_1,s_2}q^{N-1}_{s_2,s_1}\right)=\sum_{s_1=\pm1}\sum_{s_2=\pm1}\left(\sum_{s_3=\pm1}q_{s_1,s_2}q_{s_2,s_3}q^{N-2}_{s_3,s_1}\right)=\cdots\\
=&\sum_{s_1,s_2,\cdots,s_N}q_{s_1,s_2}q_{s_2,s_3}\cdots q_{s_N,s_1}\\
=&\sum_{s_1,s_2,\cdots,s_N}\prod_{i=1}^N q_{s_i,s_i+1}=\sum_{s_1,s_2,\cdots,s_N}\prod_{i=1}^N\exp\left[\frac{1}{2}h(s_i+s_{i+1})+Ks_is_{i+1}\right]\\
=&\sum_{s_1,s_2,\cdots,s_N}\exp\left[\sum_{i=1}^N(hs_i+Ks_is_{i+1})\right]\\
=&Q.
\end{aligned}
\tag{97}
$$

(b) According to the properties of trace operation, if λ_+ and λ_- are the eigenvalues of $\mathbf{q}$,

$$
Q=\mathrm{Tr}(\mathbf{q}^N)=\lambda_+^N+\lambda_-^N. \tag{98}
$$

(c) The eigenvalues of $\mathbf{q}$ are

$$
\begin{aligned}
\lambda_-=&\frac{1}{2}\left(-\sqrt{\left(e^{2h}-1\right)^2e^{-2h+2K}+4e^{-2K}}+e^{h+K}+e^{-h+K}\right)\\
=&-\sqrt{\sinh^2(h)e^{2K}+e^{-2K}}+e^K\cosh(h),
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
\lambda_+=&\frac{1}{2}\left(\sqrt{\left(e^{2h}-1\right)^2e^{-2h+2K}+4e^{-2K}}+e^{h+K}+e^{-h+K}\right)\\
=&\sqrt{\sinh^2(h)e^{2K}+e^{-2K}}+e^K\cosh(h).
\end{aligned}
\tag{100}
$$

Because $\lambda_+>\lambda_-$, in the limit $N\to\infty$,

$$
\frac{\ln Q}{N}=\ln\lambda_+=K+\ln\left[\sqrt{\sinh^2(h)+e^{-4K}}+\cosh(h)\right]. \tag{101}
$$

(d)

$$\langle s_i \rangle = \frac{\partial}{\partial h}\frac{\ln Q}{N} = \frac{\left[\sinh^2(h) + e^{-4K}\right]^{-1/2}\sinh(h)\cosh(h) + \sinh(h)}{\sqrt{\sinh^2(h) + e^{-4K}} + \cosh(h)} \tag{102}$$

$$\lim_{h\to 0^+}\langle s_i \rangle = \frac{0+0}{e^{-2K}+1} = 0. \tag{103}$$

5.22

(a) Suppose

$$Q(K,h,N) = [f(K,h)]^{N/2} Q(K',h',N/2). \tag{104}$$

After the summation of half of the s_i variables,

$$Q(K,h,N) = \sum_{s_1,s_3,s_5\cdots}\left\{\exp\left[\frac{1}{2}h(s_1+s_3+2) + K(s_1+s_3)\right] + \exp\left[\frac{1}{2}h(s_1+s_3-2) - K(s_1+s_3)\right]\right\}\cdots \tag{105}$$

Plug the expression of the partition function in,

$$\exp\left[\frac{1}{2}h(s_1+s_3+2) + K(s_1+s_3)\right] + \exp\left[\frac{1}{2}h(s_1+s_3-2) - K(s_1+s_3)\right] = f(K,h)\exp\left[\frac{1}{2}h'(s_1+s_3) + K's_1s_3\right] \tag{106}$$

$$\exp(2h+2K) + \exp(-2K) = f(K,h)\exp(h'+K') \tag{107}$$

$$\exp(2h) + \exp(-2h) = f(K,h)\exp(-K') \tag{108}$$

$$\exp(-2K) + \exp(-2h+2K) = f(K,h)\exp(-h'+K') \tag{109}$$

or equivalently

$$\ln\frac{f(K,h)}{2} - K' = \ln\cosh(2h) \tag{110}$$

$$\ln\frac{f(K,h)}{2} + h' + K' = h + \ln\cosh(h+2K) \tag{111}$$

$$\ln\frac{f(K,h)}{2} - h' + K' = -h + \ln\cosh(h-2K) \tag{112}$$

The solution is

$$\ln\frac{f(K,h)}{2} = \frac{1}{2}\ln\cosh(h) + \frac{1}{4}\ln\cosh(h+2K) + \frac{1}{4}\ln\cosh(h-2K) \tag{113}$$

$$K' = -\frac{1}{2}\ln\cosh(h) + \frac{1}{4}\ln\cosh(h+2K) + \frac{1}{4}\ln\cosh(h-2K) \tag{114}$$

$$h' = h + \frac{1}{2}\ln\cosh(h+2K) - \frac{1}{2}\ln\cosh(h-2K). \tag{115}$$

$$\therefore\ f(K,h) = 2\cosh^{1/2}(h)\cosh^{1/4}(h+2K)\cosh^{1/4}(h-2K) \tag{116}$$

$$g(K',h') = 2g(K,h) - \ln f(K,h). \tag{117}$$

(b) When $h > 0$, $h' > h$; else if $h < 0$, $h' < h$. $K' < K$ except for $h = 0$, where $K' = K$.
(c) Towards an opposite direction of flow, we need to obtain the recursion relation $K = K(K',h')$ and $h = h(K',h')$. Also,

$$g(K,h) = \frac{1}{2}g(K',h') + \frac{1}{2}\ln f(K,h) \tag{118}$$

K	h
1.0	1.0
0.46888691852368947	1.937773837047379
0.022260181862292203	2.813670795161046
7.08524842038289e-06	2.857871510136488
6.570299859723043e-13	2.857885587592979
0.0	2.857885587594285

Table 2: RG flow from high K to low K.

K	h	g
6.570299859723043e-13	2.857885587592979	2.861
1.3136158827330341e-12	2.8578855879906375	2.8610868927640976
2.6271207431567043e-12	2.8578855887856855	2.8611303395410603
5.2541304639833414e-12	2.8578855903757145	2.8611520637193366
0.000020035272038845 89	2.857845782939464	2.861143152777907
0.03638530319923742	2.7856288893140007	2.82529527237417
0.5272495462661136	1.826350352112819	2.3568108509064443
1.0424305678118018	0.9365453832540873	1.9817674540137156

Table 3: RG flow from low K to high K.

This can be done by solve the equations (114) and (115) numerically, or equivalently, solving h, K from

$$e^{4K'}\left(e^{2h}+e^{-2h}+2\right)=e^{2h}+e^{-2h}+e^{4K}+e^{-4K} \tag{119}$$

$$e^{2h'}e^{-2h}\left(e^{2h}+e^{4K}\right)=e^{2h}e^{4K}+1 \tag{120}$$

To obtain an approximate value of g(1,1), first start from $K=1.0$, $h=1.0$. We use RG equations (114) and (115). Then we start from $K=6.570299859723043e-13$ and $h=2.857885587592979$, implementing a backward RG flow. $g(6.57\times10^{-13},h)\approx g(0,h)=\ln[2\cosh(h)]$. Thus the starting $g(K,h)=2.861$.

From table 3 we know an approximate solution for $g(1,1)$ is 1.982.

5.23

(a) Since $n_i=(s_i+1)/2$, $V=N$

$$\rho=\frac{1}{N}\sum_i\langle n_i\rangle=\frac{1}{N}\sum_i\frac{\langle s_i\rangle+1}{2}=\frac{m}{2}+\frac{1}{2}. \tag{121}$$

(b) Shown in figure 4.

5.24

(a) (b) The ground state is that both two atoms are on the same side of a partition in every cell. The energy is zero and the degeneracy is 2. Another state is that two atoms are on opposite sides of a partition in every cell. The energy is $N\varepsilon$ and the degeneracy is 2^N.

(c) Obviously, the partition function is

$$Q=2+2^Ne^{-\beta N\varepsilon}. \tag{122}$$

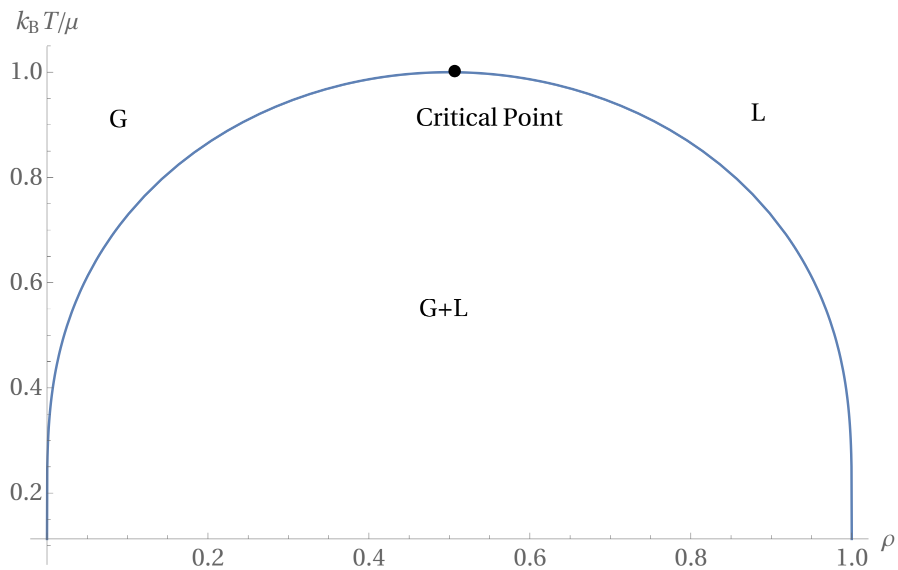

Figure 4: T-ρ coexistence curve of the 2D lattice gas model from the mean field theory.

(d) The free energy per particle is

$$a = -\frac{1}{N\beta}\ln Q = -\frac{1}{N\beta}\ln\left[2 + \left(\frac{2}{e^{\beta\varepsilon}}\right)^N\right] \tag{123}$$

When $e^{\beta\varepsilon} > 2$, or $\beta\varepsilon > \ln 2$, as $N \to \infty$, $a \to -\frac{1}{N\beta}\ln 2 \to 0$. When $\beta\varepsilon < 2$, as $N \to \infty$, $a \to -\beta^{-1}\ln 2 + \varepsilon$. Because

$$\langle E\rangle = \frac{\partial \ln Q}{\partial(-\beta)} = \frac{\partial \beta A}{\partial \beta}, \tag{124}$$

at $\beta_0 = \ln 2/\varepsilon$, $\langle E\rangle$ is discontinuous.

(e)

$$T_0 = \frac{\varepsilon}{k_B \ln 2}. \tag{125}$$

5.25

For the Ising model under zero field, the average total energy from mean field theory is

$$\langle E\rangle = -\frac{1}{2}JNzm^2 \tag{126}$$

When $\beta Jz < 1$, $m = 0$. $\langle E\rangle = 0$. Thus $C = 0$. When $\beta Jz > 1$, m is decided by β

$$\beta = \frac{1}{2Jzm}\ln\left(\frac{1+m}{1-m}\right) \tag{127}$$

Notice that when $\beta Jz \to 1^+$, $m \to 0^+$.

$$\frac{d\beta}{dm} = -\frac{1}{2Jzm^2}\ln\left(\frac{1+m}{1-m}\right) + \frac{1}{Jzm(1-m^2)}. \tag{128}$$

$$
\begin{aligned}
C =& -k_B\beta^2\frac{\partial\langle E\rangle}{\partial\beta} = k_B\beta^2 JNzm\frac{dm}{d\beta}\\
=&\frac{1}{4}k_B\frac{1}{J^2z^2m^2}\ln^2\left(\frac{1+m}{1-m}\right)JNzmJzm\left[-\frac{1}{2m}\ln\left(\frac{1+m}{1-m}\right)+\frac{1}{1-m^2}\right]^{-1}\\
=&\frac{1}{4}Nk_B\ln^2\left(\frac{1+m}{1-m}\right)\left[-\frac{1}{m}\ln\left(\frac{1+m}{1-m}\right)+\frac{1}{1-m^2}\right]^{-1}
\end{aligned}
\tag{129}
$$

$$
\begin{aligned}
\lim_{\beta Jz\to 1^+} C = \lim_{m\to 0^+} C =& \frac{1}{4}Nk_B\lim_{m\to 0^+}\ln^2\left(\frac{1+m}{1-m}\right)\left[-\frac{1}{m}\ln\left(\frac{1+m}{1-m}\right)+\frac{1}{1-m^2}\right]^{-1}\\
=&\frac{1}{4}Nk_B\times 6 = \frac{3}{2}Nk_B.
\end{aligned}
\tag{130}
$$

In conclusion, the heat capacity just below the critical temperature is $\frac{3}{2}Nk_B$, and just above the critical temperature is 0.

5.26

(a)

$$
\begin{aligned}
Q =& \int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma}Q(\mathscr{E})\\
=&\int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma}\lim_{P\to\infty}\sum_{\{u_i\}}(\varepsilon\Delta)^{P/2}\exp\left[\sum_{i=1}^{P}(\kappa u_i u_{i+1}+\varepsilon\mu\mathscr{E}u_i)\right]\\
=&\lim_{P\to\infty}\sum_{\{u_i\}}(\varepsilon\Delta)^{P/2}\exp\left[\sum_{i=1}^{P}(\kappa u_i u_{i+1})\right]\int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma}\exp\left[\sum_{i=1}^{P}(\varepsilon\mu\mathscr{E}u_i)\right]\\
=&\lim_{P\to\infty}\sum_{\{u_i\}}(\varepsilon\Delta)^{P/2}\exp\left[\sum_{i=1}^{P}(\kappa u_i u_{i+1})\right]\sqrt{\frac{2\pi\sigma}{\beta}}\exp\left[\frac{\sigma}{2\beta}\left(\sum_{i=1}^{P}(\varepsilon\mu u_i)\right)^2\right]\\
=&\sqrt{\frac{2\pi\sigma}{\beta}}\lim_{P\to\infty}\sum_{\{u_i\}}(\varepsilon\Delta)^{P/2}\exp\left[\sum_{i=1}^{P}(\kappa u_i u_{i+1})\right]\exp\left[\frac{\sigma}{2\beta}\left(\sum_{i=1}^{P}(\varepsilon\mu u_i)\right)^2\right]\\
=&\sqrt{\frac{2\pi\sigma}{\beta}}\lim_{P\to\infty}\left\{(\varepsilon\Delta)^{P/2}\sum_{\{u_i\}}\exp\left[\sum_{i=1}^{P}\kappa u_i u_{i+1}+\frac{\beta\mu^2\sigma}{2P^2}\sum_{i,j=1}^{P}u_i u_j\right]\right\}
\end{aligned}
\tag{131}
$$

(b) (i) Evaluating $Q(\mathscr{E})$. Using the result of the isomorphism to a quantum two-level system, which is introduced in section 5.8 in the textbook,

$$
\begin{aligned}
Q(\mathscr{E}) =& \lim_{P\to\infty}\sum_{\{u_i\}}(\varepsilon\Delta)^{P/2}\exp\left[\sum_{i=1}^{P}(\kappa u_i u_{i+1}+\varepsilon\mu\mathscr{E}u_i)\right]\\
=&2\cosh\left(\beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right).
\end{aligned}
\tag{132}
$$

(ii) Show the Gaussian weighted integral over $\mathscr{E}$ that gives Q is a non-analytic function of σ when $\beta \to \infty$.

$$\begin{aligned} m = \mu\langle u_1\rangle &= \int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma} \frac{1}{Q}\frac{\partial Q(\mathscr{E})}{\partial(\beta\mathscr{E})} \\ &= \mu \frac{\int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma} 2\sinh\left(\beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)\mu\mathscr{E}/\sqrt{\Delta^2+\mu^2\mathscr{E}^2}}{\int_{-\infty}^{\infty} d\mathscr{E} e^{-\beta\mathscr{E}^2/2\sigma} 2\cosh\left(\beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)} \end{aligned} \tag{133}$$

When $\beta \to \infty$,

$$m = \mu \frac{\int_{-\infty}^{\infty} d\mathscr{E} \exp\left(-\beta\mathscr{E}^2/2\sigma + \beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)\mu\mathscr{E}/\sqrt{\Delta^2+\mu^2\mathscr{E}^2}}{\int_{-\infty}^{\infty} d\mathscr{E} \exp\left(-\beta\mathscr{E}^2/2\sigma + \beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)} \tag{134}$$

Define the function

$$w(\mathscr{E}) = \frac{\sqrt{\beta}\exp\left(-\beta\mathscr{E}^2/2\sigma + \beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)}{\int_{-\infty}^{\infty} d\mathscr{E}\sqrt{\beta}\exp\left(-\beta\mathscr{E}^2/2\sigma + \beta\sqrt{\Delta^2+\mu^2\mathscr{E}^2}\right)} \tag{135}$$

Then

$$m = \mu\int_{-\infty}^{\infty} d\mathscr{E} w(\mathscr{E}) \frac{\mu\mathscr{E}}{\sqrt{\Delta^2+\mu^2\mathscr{E}^2}} \tag{136}$$

When $\beta \to \infty$, $w(\mathscr{E})$ becomes a delta function. It peaks at $\mathscr{E}_\pm$,

$$\mathscr{E}_\pm = \pm\sqrt{\sigma^2\mu^2 - \frac{\Delta^2}{\mu^2}} \tag{137}$$

if $\sigma^2\mu^2 > \frac{\Delta^2}{\mu^2}$, namely $\dfrac{\sigma\mu^2}{\Delta} > 1$. Then $m \neq zero$. Else if $\dfrac{\sigma\mu^2}{\Delta} \leqslant 1$, $m = 0$.

5.27

(a) Using the result of Exercise 5.21,

$$Q(h;\beta,N) = e^{-\beta Nh^2/2\sigma}\exp\left(N\left[\beta J + \ln\left\{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right\}\right]\right) \tag{138}$$

$$\tilde{A}(h;\beta,N) = -\beta^{-1}\ln Q(h;\beta,N) = Nh^2/2\sigma - N\left[J + \frac{1}{\beta}\ln\left\{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right\}\right]. \tag{139}$$

(b)

$$\frac{\partial\tilde{A}(h;\beta,N)}{\partial h} = N\left[\frac{h}{\sigma} - \frac{\sinh(\beta h) + \frac{\sinh(\beta h)\cosh(\beta h)}{\sqrt{\sinh^2(\beta h)+e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}\right] = 0 \tag{140}$$

$$\frac{h}{\sigma}\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right] = \sinh(\beta h) + \frac{\sinh(\beta h)\cosh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} \tag{141}$$

Let

$$f(h) = \frac{\sinh(\beta h) + \frac{\sinh(\beta h)\cosh(\beta h)}{\sqrt{\sinh^2(\beta h)+e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} \tag{142}$$

$$f'(0) = \beta e^{-2\beta J} \tag{143}$$

According to figure of the function $f(h)$, if $f'(0) > \frac{1}{\sigma}$, then h has two non-zero solutions. Because when given σ and J as constants,

$$\sigma\beta e^{2\beta J} > 1 \tag{144}$$

gives $\beta >$ constant. Therefore above some value of β, the free energy becomes a bistable function of h.
(c) The critical temperature should satisfy

$$k_B T_c = \sigma \exp\left(\frac{2J}{k_B T_c}\right). \tag{145}$$